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# HEAT TRANSFER TO A NEAR-SEPARATING LAMINAR BOUNDARY LAYER

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**Abstract**—Two aspects of heat transfer to a laminar boundary layer are discussed. First, an exact solution to the thermal-energy equation is given for a flow, with continuously zero wall shear stress, subjected to a steep change in wall temperature. The principle of superposition is used to obtain an expression for the heat-transfer rate for a power-law variation of surface to free-stream temperature difference. The second part of this note concerns the use of the above result, for the case of uniform surface temperature, to improve Spalding's [1] refinement of Lighthill's [2] general method for the calculation of heat transfer through laminar boundary layers. Spalding's correction function is extended further into the regime of extreme adverse pressure gradient, and an analytical formula given for its asymptotic behavior at the separation point.

## NOMENCLATURE

$F$, function of $\lambda$ defined by Spalding [1];
$h$, local surface heat-transfer coefficient [W/m$^2$°C];
$k$, thermal conductivity of fluid [W/m°C];
$m$, exponent of $x^{-1}$ for power-law variation of $u_\infty$;
$n$, exponent of $x$ for power-law variation of $T_s - T_\infty$;
$Nu$, Nusselt number;
$q_s$, surface heat flux [W/m$^2$];
$Re$, Reynolds number;
$T$, fluid temperature [°C];
$T_s$, surface temperature (value of $T$ at $y = 0$) [°C];
$T_\infty$, mainstream temperature (value of $T$ for $y \to \infty$) [°C];
$u$, component of fluid velocity in $x$-direction [m/s];
$u_\infty$, mainstream velocity (value of $u$ for $y \to \infty$) [m/s];
$v$, component of fluid velocity in $y$-direction [m/s];
$x$, distance measured along surface [m];
$y$, distance measured normal to surface [m];
$\alpha$, thermal diffusivity of fluid [m$^2$/s];
$\delta$, boundary-layer length scale [m];
$\eta$, non-dimensional length measured normal to surface;
$\lambda$, argument of Spalding's correction function $F$;
$\mu$, dynamic viscosity of fluid [P];
$\nu$, kinematic viscosity of fluid [m$^2$/s];
$\xi$, unheated starting length [m];
$\sigma$, Prandtl number;
$\tau_s$, surface shear stress [N/m$^2$].

## 1. INTRODUCTION

LIGHTHILL [2] observed that so far as heat transfer through a laminar boundary layer is concerned, the crucial region of the velocity profile is that close to the surface where, if the shear stress is non-zero, the velocity $u$ varies linearly with $y$, the normal distance from the wall. On this basis, Lighthill found an exact solution of the thermal-energy equation and deduced from it the expression below for the surface heat-transfer coefficient $h$

$$h(x) = \frac{\tau_s^{\frac{1}{2}} k}{\Gamma(4/3)3^{\frac{2}{3}}} \left( \int_0^x \tau_s^{\frac{1}{2}} \,\mathrm{d}x \right)^{-\frac{1}{3}}. \tag{1}$$

The surface shear stress $\tau_s(x)$ is presumed known.

For a boundary layer with zero wall shear stress, Lighthill's analysis fails because the leading term in the series expansion for $u(y)$ is quadratic rather than linear. The first part of this note is concerned with a solution procedure, similar to Lighthill's, for a flow with $\tau_s$ continuously zero.

Spalding [1] added a correction term to Lighthill's result to allow for departures from linearity of the velocity profile in the vicinity of the surface, for both favorable and adverse pressure gradients. The correction function was determined primarily on the basis of isothermal-wedge solutions, but was not continued all the way to the separation point. The second part of this note shows how the result of the first part can be used to obtain an approximate analytical form for Spalding's function in the vicinity of the separation point. This approximation is then used, together with data from the wedge-flow solutions of Evans [3], to improve the accuracy of the correction term in the severe adverse pressure-gradient region, and to extend it up to the separation point.

## 2. LAMINAR BOUNDARY LAYER WITH $\tau_s = 0$

(a) $T_s$ *constant*, $x \geqslant \xi$

The thermal-energy equation for a laminar boundary with uniform fluid properties and negligible viscous dissipation is

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}, \tag{2}$$

with boundary conditions, for an unheated starting length $\xi$

$$y = 0, \quad x \geqslant \xi: \qquad T = T_s$$

$$y \to \infty: \qquad T = T_\infty, \quad \frac{\partial T}{\partial y} = 0.$$

For a laminar boundary layer with continuously zero wall shear stress (a wedge flow with $u_\infty \sim x^{-m}$, $m = 0{\cdot}0904$), the velocity components $u$ and $v$, close to the surface, are given by

$$u = \frac{mu_\infty^2 y^2}{2vx} \quad \text{and} \quad v = \frac{m(2m+1)u_\infty^2 y^3}{6vx}. \tag{3}$$

Provided the thermal boundary layer is thin compared with the viscous boundary layer, these asymptotic expressions for $u$ and $v$ may be substituted into (2). Then, if $T(x, y)$ (for $x \geqslant \xi$) is assumed to be a function of the similarity variable $\eta \equiv y/\delta$, where $\delta(x)$ is a length scale to be determined, (2) reduces to the ordinary differential equation

$$T'' + \frac{mu_\infty^2\delta^3}{2v\alpha x}\left[\frac{d\delta}{dx} - \frac{(2m+1)}{3}\frac{\delta}{x}\right]\eta^3 T' = 0.$$

The similarity assumption is seen to be valid provided only that the coefficient of $\eta^3 T'$ is a constant, which may be chosen arbitrarily and for convenience is taken as unity. Then the equation for $T$ reduces to

$$T'' + \eta^3 T' = 0$$

for which the solution is

$$\frac{T_s - T}{T_s - T_\infty} = \frac{\gamma\left(\frac{1}{4}, \frac{\eta^4}{4}\right)}{\Gamma(\frac{1}{4})} \tag{4}$$

where $\Gamma(\frac{1}{4}) = \gamma(\frac{1}{4}, \infty)$ is a Gamma function, and

$$\gamma\left(\frac{1}{4}, \frac{\eta^4}{4}\right) \equiv \int_0^{\frac{\eta^4}{4}} z^{-\frac{3}{4}} e^{-z} dz$$

an incomplete Gamma function.

The arbitrariness in the choice for the constant in the equation for $T(\eta)$ is a direct result of the arbitrary nature of the thickness of the thermal boundary layer, to which the length scale $\delta$ is proportional. The differential equation for $\delta$ is

$$\frac{mu_\infty^2\delta^3}{2v\alpha x}\left[\frac{d\delta}{dx} - \frac{(2m+1)}{3}\frac{\delta}{x}\right] = 1$$

which may be rewritten in a form more convenient for integration as

$$\frac{d}{dx}\left[\delta^4 x^{-4(2m+1)/3}\right] = \frac{8v\alpha}{mu_\infty^2} x^{-(8m+1)/3} \tag{5}$$

subject to the boundary condition

$$x = \xi, \qquad \delta = 0.$$

The solution for $\delta$ is

$$\delta = \left[\frac{12}{m(1-m)\sigma}\right]^{\frac{1}{4}}\left(\frac{vx}{u_\infty}\right)^{\frac{1}{2}}\left[1 - (\xi/x)^{2(1-m)/3}\right]^{\frac{1}{4}}, \tag{6}$$

where $\sigma \equiv v/\alpha$ is the Prandtl number.

Then the following expression for the local heat transfer coefficient $h \equiv q_s/(T_s - T_\infty)$ is easily obtained

$$h(x, \xi) = \frac{[m(1-m)\sigma/3]^{\frac{1}{4}}}{2\Gamma(\frac{5}{4})} k\left(\frac{u_\infty}{vx}\right)^{\frac{1}{2}} \times \left[1 - (\xi/x)^{2(1-m)/3}\right]^{-\frac{1}{4}} \tag{7}$$

or, in non-dimensional form, and with $m = 0{\cdot}0904$

$$Nu\,Re^{-\frac{1}{2}}\sigma^{-\frac{1}{4}} = 0{\cdot}225\left[1 - (\xi/x)^{0{\cdot}604}\right]^{-\frac{1}{4}} \tag{8}$$

where $Nu \equiv hx/k$ and $Re \equiv u_\infty x/v$. A similar result, for $\xi = 0$, was derived by Liepmann [4] using an integral analysis.

(b) $T_s(x)$ *arbitrarily specified*

If the surface temperature $T_s$ varies with $x$ in a known way, then the local heat transfer rate $q_s$ at any point $x$ can be determined by the method of superposition, through evaluation of the Stieltjes integral

$$q_s = \int_0^x h(x, \xi)\, dT_s(\xi), \tag{9}$$

wherein $h(x, \xi)$ is given by (7). In the case of a power-law variation of $T_s - T_\infty$ with $x$, i.e.

$$T_s - T_\infty \sim x^n,$$

the Stieltjes integral can be evaluated in a straightforward manner as a Rieman integral, leading to the following result in terms of a Beta function

$$Nu\,Re^{-\frac{1}{2}}\sigma^{-\frac{1}{4}} = 0{\cdot}225\frac{3n}{2(1-m)}\beta\left[\frac{3}{4}, \frac{3n}{2(1-m)}\right]. \tag{10}$$

The value of $m$ is 0·0904 as before.

## 3. EXTENSION OF THE LIGHTHILL–SPALDING CALCULATION METHOD TO $\tau_s = 0$

Although his formula (1) for $h$ was proposed for general use, Lighthill [2] indicated that it becomes increasingly inaccurate as separation is approached, essentially because the linear region of the velocity profile then diminishes to vanishing point. As mentioned in the introduction, Spalding [1] added a correction term to Lighthill's result to allow for departures from linearity of the velocity profile, for both favorable and adverse pressure gradients. Spalding's modified form of (1) is most conveniently written in differential form as

$$\frac{k^3}{\alpha\mu\tau_s^{\frac{1}{2}}}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\tau_s^{\frac{3}{2}}}{h^3}\right) = 9\Gamma(\tfrac{4}{3})^3 + F(\lambda) \qquad (11)$$

where

$$\lambda \equiv \frac{\rho k u_\infty}{h\tau_s}\frac{\mathrm{d}u_\infty}{\mathrm{d}x}.$$

The correction function $F(\lambda)$ was determined primarily on the basis of the isothermal-wedge solutions, for $0{\cdot}7 < \sigma < 10$, and given in graphical form. However, Spalding did not specify $F(\lambda)$ for $\lambda < -7$ (approximately) so that (11) cannot be employed, as it stands, all the way to the separation point. The purpose of the second part of this note is to show that an approximate analytical form for $F(\lambda)$ for $-\lambda \gg 1$ may be deduced from (7), thus permitting continuous specification of $F(\lambda)$ up to and including $\lambda = -\infty$, the separation point.

For isothermal-wedge flows, the term on the left hand side of (11) may be simplified to

$$\tfrac{3}{4}(1 - m)\frac{k^3\tau_s}{\mu\alpha h^3 x}.$$

Also, as $\tau_s \to 0$, $1/\lambda \to 0$ and $F(\lambda)$ may be expanded as a power series in $1/\lambda$, i.e.

$$F(\lambda) = F(\infty) + \frac{1}{\lambda}F'(\infty) + 0(1/\lambda^2),$$

so that (11) becomes

$$\tfrac{3}{4}(1 - m)\frac{k^3\tau_s}{\mu\alpha h^3 x} \approx 6{\cdot}41 + F(\infty) + \frac{h\tau_s}{\rho k u_\infty(\mathrm{d}u_\infty/\mathrm{d}x)}F'(\infty).$$

Evidently $F(\infty) = -6{\cdot}41$, if the latter result is to hold for $\tau_s = 0$, and

$$F'(\infty) = \tfrac{3}{4}(1 - m)\left(\frac{k}{h}\right)^4 \frac{u_\infty}{\alpha\nu x}\frac{\mathrm{d}u_\infty}{\mathrm{d}x}\bigg|_{\tau_s = 0} = 36\Gamma(\tfrac{5}{4})^4 = 23{\cdot}4$$

(from (7) with $\xi = 0$). Then, for $-\lambda \gg 1$, we have

$$F(\lambda) + 6{\cdot}41 \approx 23{\cdot}4/\lambda. \qquad (12)$$

In Fig. 1, this result is used, together with data from the numerical solutions of Evans [3] for isothermal wedges, to extend the curve for $F(\lambda)$ to $-1/\lambda = 0$ ($\tau_s = 0$). For consistency with Spalding's analysis, only data points with $0{\cdot}7 < \sigma < 10$ have been plotted. Two defects in Spalding's curve for $F(\lambda)$ are removed in Fig. 1. First, the curve is extended beyond $\lambda = -7{\cdot}5$; secondly, for $-1/\lambda < 0{\cdot}35$ Spalding's original curve is in error, lying somewhat above the broken curve of

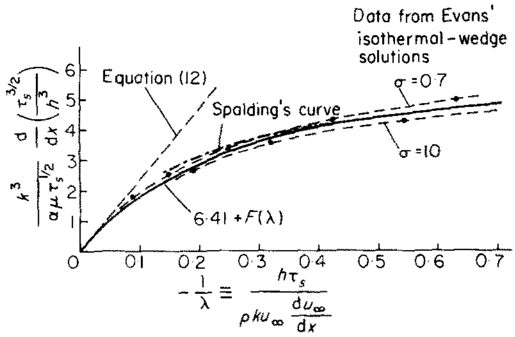


FIG. 1. Extension of Spalding's $F(\lambda)$ function to $1/\lambda = 0$.

Fig. 1 for $\sigma = 0{\cdot}7$. The data of Evans, used here, presumably are both more extensive and reliable than those available to Spalding in 1958.

## 4. CONCLUDING REMARKS

Through an exact solution of the thermal-energy equation, an expression was obtained for the heat-transfer coefficient for a laminar boundary layer with continuously zero wall shear stress and arbitrary variation of surface temperature.

This result, together with the numerical solutions of Evans for isothermal wedge flows, was used to improve the accuracy of Spalding's refinement of the Lighthill method for calculating heat transfer in laminar flow, and to extend its range of application to include the separation point.

## REFERENCES

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## TRANSFERT THERMIQUE POUR UNE COUCHE LIMITE PROCHE DU DÉCOLLEMENT

**Résumé**—On discute deux aspects du transfert thermique dans une couche limite laminaire. Tout d'abord on donne une solution exacte de l'équation de l'énergie pour un écoulement à tension pariétale continûment nulle et qui est soumis à un échelon de température de paroi. On utilise la principe de superposition pour obtenir l'expression du flux thermique dans le cas d'une variation en loi puissance de la différence de température entre surface et fluide au loin. La second partie de cette note applique les résultats obtenus au cas de la température de surface uniforme afin de pousser plus loin l'amélioration, due à Spalding, de la méthode générale de Lighthill pour le calcul du transfert thermique à travers les couches limites laminaires. Les fonctions de correction de Spalding sont étendues au régime des forts gradients de pression adverse et une formule analytique est donnée pour le comportement asymptotique au point de séparation.

## DER WÄRMEÜBERGANG AN EINE LAMINARE GRENZSCHICHT NAHE IHRER ABLÖSUNG

**Zusammenfassung**—Es werden zwei Gesichtspunkte des Wärmeübergangs an eine laminare Grenzschicht diskutiert. Zuerst wird eine exakte Lösung der thermischen Energiegleichung für eine Strömung mit verschwindender Wandschubspannung und einem plötzlichen Sprung in der Wandtemperatur gegeben. Mit Hilfe des Superpositionsprinzips wurde ein Ausdruck für den Wärmeübergang bei einer Veränderung der Temperaturdifferenz zwischen Wand und Freistrom nach einem Potenzgesetz ermittelt.

Im zweiten Teil werden obige Ergebnisse verwertet für den Fall gleichförmiger Oberflächentemperatur, um Spaldings [1] Verfeinerung der allgemeinen Methode von Lighthill [2] für die Berechnung des Wärmeüberganges durch laminare Grenzschichten weiter zu verbessern. Die Korrekturfunktion nach Spalding wurde ausgedehnt in das Gebiet extremer gegenläufiger Druckgradienten und es wird eine Formel für deren asymptotisches Verhalten beim Ablösungspunkt angegeben.

## ПЕРЕНОС ТЕПЛА К РАЗДЕЛЯЮЩЕМУ ЛАМИНАРНОМУ ПОГРАНИЧНОМУ СЛОЮ

**Аннотация**—Обсуждаются два аспекта переноса тепла в ламинарный пограничный слой. Сначала приводится точное решение уравнения тепловой энергии для потока с постоянным нулевым напряжением сдвига на стенке при скачкообразном изменении температуры стенки. Принцип суперпозиции применяется для получения выражения скорости переноса тепла при степенной зависимости разности температур поверхности и свободного потока. Во второй части рассматривается использование полученных результатов для поверхности с равномерной температурой с целью дальнейшего развития поправки Сполдинга к общему методу расчета переноса тепла через ламинарный пограничный слой, разработанному Лайтхиллом. Поправочная функция Сполдинга обобщается для режима предельного положительного градиента давления. Приводится аналитическая формула, выражающая асимптотическое поведение течения в точке отрыва.